

$n = 9;$

$$\gamma_{\text{sol}} = \frac{e^{c[1]} \left((-1 + e^{c[2]}) c[1] - c[2] \right) + c[2]}{(-1 + e^{c[1]}) c[2] (c[1] + c[2])};$$

$\text{Series}[\gamma_{\text{sol}} /. c[i_] \Rightarrow c[i] h, \{h, 0, n\}]$

$$\begin{aligned} & \frac{1}{2} + \frac{1}{12} (c[1] + 2 c[2]) h + \frac{1}{24} c[2] (c[1] + c[2]) h^2 + \\ & \frac{1}{720} (-c[1]^3 + c[1]^2 c[2] + 9 c[1] c[2]^2 + 6 c[2]^3) h^3 + \\ & \frac{c[2] (-c[1]^3 + c[1]^2 c[2] + 4 c[1] c[2]^2 + 2 c[2]^3) h^4}{1440} + \\ & \frac{(c[1]^5 - c[1]^4 c[2] - 6 c[1]^3 c[2]^2 + 6 c[1]^2 c[2]^3 + 15 c[1] c[2]^4 + 6 c[2]^5) h^5}{30240} + \\ & \frac{c[2] (2 c[1]^5 - 2 c[1]^4 c[2] - 5 c[1]^3 c[2]^2 + 5 c[1]^2 c[2]^3 + 9 c[1] c[2]^4 + 3 c[2]^5) h^6}{120960} + \\ & \frac{1}{3628800} (-3 c[1]^7 + 3 c[1]^6 c[2] + 17 c[1]^5 c[2]^2 - 17 c[1]^4 c[2]^3 - \\ & \quad 25 c[1]^3 c[2]^4 + 25 c[1]^2 c[2]^5 + 35 c[1] c[2]^6 + 10 c[2]^7) h^7 + \\ & \frac{1}{7257600} c[2] (-3 c[1]^7 + 3 c[1]^6 c[2] + 7 c[1]^5 c[2]^2 - 7 c[1]^4 c[2]^3 - \\ & \quad 7 c[1]^3 c[2]^4 + 7 c[1]^2 c[2]^5 + 8 c[1] c[2]^6 + 2 c[2]^7) h^8 + \frac{1}{239500800} \\ & (5 c[1]^9 - 5 c[1]^8 c[2] - 28 c[1]^7 c[2]^2 + 28 c[1]^6 c[2]^3 + 38 c[1]^5 c[2]^4 - \\ & \quad 38 c[1]^4 c[2]^5 - 28 c[1]^3 c[2]^6 + 28 c[1]^2 c[2]^7 + 27 c[1] c[2]^8 + 6 c[2]^9) h^9 + O[h]^{10} \end{aligned}$$

$\text{eqns1} =$

$$\begin{aligned} & \left\{ (-e^{c[1]} (c[1] + c[2]) (1 + \alpha c[1] + \gamma c[2]) (1 + \beta c[1] + \delta c[2]) + c[2] (1 + \delta c[2] + \alpha c[1] \right. \\ & \quad (1 + \delta c[2]) + \gamma (c[1] + \beta c[1]^2 + c[2] + \delta c[1] c[2] + \delta c[2]^2)) + e^{c[1]+c[2]} c[1] \\ & \quad (1 + \delta c[2] + \beta (c[1] + c[2] + \gamma c[2]^2) + \alpha c[1] (1 + \delta c[2] + \beta (c[1] + c[2]))) \left. \right) / \\ & (c[1] (c[1] + c[2]) (1 + \alpha c[1] + \gamma c[2]) (1 + \beta c[1] + \delta c[2])) = 0, \\ & (e^{c[1]} (c[1] + c[2]) (1 + \alpha c[1] + \gamma c[2]) (1 + \beta c[1] + \delta c[2]) - c[2] (1 + \delta c[2] + \\ & \quad \alpha c[1] (1 + \delta c[2]) + \gamma (c[1] + \beta c[1]^2 + c[2] + \delta c[1] c[2] + \delta c[2]^2)) - e^{c[1]+c[2]} c[1] \\ & \quad (1 + \delta c[2] + \beta (c[1] + c[2] + \gamma c[2]^2) + \alpha c[1] (1 + \delta c[2] + \beta (c[1] + c[2]))) \left. \right) / \\ & (c[2] (c[1] + c[2]) (1 + \alpha c[1] + \gamma c[2]) (1 + \beta c[1] + \delta c[2])) = 0 \} /. \{\alpha | \beta | \delta \rightarrow 0\} \\ & \left\{ \frac{e^{c[1]+c[2]} c[1] - e^{c[1]} (c[1] + c[2]) (1 + \gamma c[2]) + c[2] (1 + \gamma (c[1] + c[2]))}{c[1] (c[1] + c[2]) (1 + \gamma c[2])} = 0, \right. \\ & \left. \frac{-e^{c[1]+c[2]} c[1] + e^{c[1]} (c[1] + c[2]) (1 + \gamma c[2]) - c[2] (1 + \gamma (c[1] + c[2]))}{c[2] (c[1] + c[2]) (1 + \gamma c[2])} = 0 \right\} \end{aligned}$$

Reduce[eqns1, γ]

$$\left((-1 + e^{c[1]}) c[2] (c[1] + c[2]) \neq 0 \ \&\& \right. \\ \left. (-1 + e^{c[1]+c[2]}) c[1] \neq 0 \ \&\& \gamma = \frac{-e^{c[1]} c[1] + e^{c[1]+c[2]} c[1] + c[2] - e^{c[1]} c[2]}{(-1 + e^{c[1]}) c[2] (c[1] + c[2])} \right) \parallel \\ ((C[1] | C[2]) \in \text{Integers} \ \&\& c[1] c[2] (c[1] + c[2]) (1 + \gamma c[2]) \neq 0 \ \&\& \\ c[1] = 2 i \pi C[2] \ \&\& c[2] = -c[1] + 2 i \pi C[1])$$

$\gamma_s = \text{Sum}[\gamma_{10}^{i+j} c[1]^i c[2]^j h^{i+j}, \{i, 0, n\}, \{j, 0, n\}] + O[h]^{(n+1)}$

$$\gamma_0 + (c[2] \gamma_1 + c[1] \gamma_{10}) h + (c[2]^2 \gamma_2 + c[1] c[2] \gamma_{11} + c[1]^2 \gamma_{20}) h^2 + \\ (c[2]^3 \gamma_3 + c[1] c[2]^2 \gamma_{12} + c[1]^2 c[2] \gamma_{21} + c[1]^3 \gamma_{30}) h^3 + \\ (c[2]^4 \gamma_4 + c[1] c[2]^3 \gamma_{13} + c[1]^2 c[2]^2 \gamma_{22} + c[1]^3 c[2] \gamma_{31} + c[1]^4 \gamma_{40}) h^4 + \\ (c[2]^5 \gamma_5 + c[1] c[2]^4 \gamma_{14} + c[1]^2 c[2]^3 \gamma_{23} + c[1]^3 c[2]^2 \gamma_{32} + c[1]^4 c[2] \gamma_{41} + c[1]^5 \gamma_{50}) h^5 + \\ (c[2]^6 \gamma_6 + c[1] c[2]^5 \gamma_{15} + c[1]^2 c[2]^4 \gamma_{24} + c[1]^3 c[2]^3 \gamma_{33} + c[1]^4 c[2]^2 \gamma_{42} + \\ c[1]^5 c[2] \gamma_{51} + c[1]^6 \gamma_{60}) h^6 + (c[2]^7 \gamma_7 + c[1] c[2]^6 \gamma_{16} + c[1]^2 c[2]^5 \gamma_{25} + \\ c[1]^3 c[2]^4 \gamma_{34} + c[1]^4 c[2]^3 \gamma_{43} + c[1]^5 c[2]^2 \gamma_{52} + c[1]^6 c[2] \gamma_{61} + c[1]^7 \gamma_{70}) h^7 + \\ (c[2]^8 \gamma_8 + c[1] c[2]^7 \gamma_{17} + c[1]^2 c[2]^6 \gamma_{26} + c[1]^3 c[2]^5 \gamma_{35} + c[1]^4 c[2]^4 \gamma_{44} + \\ c[1]^5 c[2]^3 \gamma_{53} + c[1]^6 c[2]^2 \gamma_{62} + c[1]^7 c[2] \gamma_{71} + c[1]^8 \gamma_{80}) h^8 + \\ (c[2]^9 \gamma_9 + c[1] c[2]^8 \gamma_{18} + c[1]^2 c[2]^7 \gamma_{27} + c[1]^3 c[2]^6 \gamma_{36} + c[1]^4 c[2]^5 \gamma_{45} + \\ c[1]^5 c[2]^4 \gamma_{54} + c[1]^6 c[2]^3 \gamma_{63} + c[1]^7 c[2]^2 \gamma_{72} + c[1]^8 c[2] \gamma_{81} + c[1]^9 \gamma_{90}) h^9 + O[h]^{10}$$

eqns2 = (eqns1 /. { $\gamma \rightarrow \gamma_s$, $c[i_] \rightarrow hc[i]$ } // Simplify // Normal // Collect[#, h] &);

vars = Union[Cases[eqns2, γ , Infinity]];

psol = SolveAlways[eqns2, {c[1], c[2], h}]

$$\left\{ \left\{ \gamma_{26} \rightarrow \frac{1}{1036800}, \gamma_{35} \rightarrow -\frac{1}{1036800}, \gamma_{44} \rightarrow -\frac{1}{1036800}, \gamma_{53} \rightarrow \frac{1}{1036800}, \gamma_{62} \rightarrow \frac{1}{2419200}, \right. \right. \\ \left. \gamma_{71} \rightarrow -\frac{1}{2419200}, \gamma_8 \rightarrow \frac{1}{3628800}, \gamma_{17} \rightarrow \frac{1}{907200}, \gamma_{80} \rightarrow 0, \gamma_{25} \rightarrow \frac{1}{145152}, \gamma_{34} \rightarrow -\frac{1}{145152}, \right. \\ \left. \gamma_{43} \rightarrow -\frac{17}{3628800}, \gamma_{52} \rightarrow \frac{17}{3628800}, \gamma_{61} \rightarrow \frac{1}{1209600}, \gamma_{70} \rightarrow -\frac{1}{1209600}, \gamma_{16} \rightarrow \frac{1}{103680}, \right. \\ \left. \gamma_7 \rightarrow \frac{1}{362880}, \gamma_{24} \rightarrow \frac{1}{24192}, \gamma_{33} \rightarrow -\frac{1}{24192}, \gamma_{42} \rightarrow -\frac{1}{60480}, \gamma_{51} \rightarrow \frac{1}{60480}, \gamma_{60} \rightarrow 0, \right. \\ \left. \gamma_{15} \rightarrow \frac{1}{13440}, \gamma_6 \rightarrow \frac{1}{40320}, \gamma_{23} \rightarrow \frac{1}{5040}, \gamma_{32} \rightarrow -\frac{1}{5040}, \gamma_{41} \rightarrow -\frac{1}{30240}, \gamma_{50} \rightarrow \frac{1}{30240}, \right. \\ \left. \gamma_{14} \rightarrow \frac{1}{2016}, \gamma_5 \rightarrow \frac{1}{5040}, \gamma_{22} \rightarrow \frac{1}{1440}, \gamma_{31} \rightarrow -\frac{1}{1440}, \gamma_{40} \rightarrow 0, \gamma_{13} \rightarrow \frac{1}{360}, \gamma_4 \rightarrow \frac{1}{720}, \gamma_{21} \rightarrow \frac{1}{720}, \right. \\ \left. \gamma_{30} \rightarrow -\frac{1}{720}, \gamma_{12} \rightarrow \frac{1}{80}, \gamma_3 \rightarrow \frac{1}{120}, \gamma_{20} \rightarrow 0, \gamma_{11} \rightarrow \frac{1}{24}, \gamma_2 \rightarrow \frac{1}{24}, \gamma_{10} \rightarrow \frac{1}{12}, \gamma_1 \rightarrow \frac{1}{6}, \gamma_0 \rightarrow \frac{1}{2} \right\} \}$$

`ys /. psol // Simplify`

$$\begin{aligned} & \left\{ \frac{1}{2} + \frac{1}{12} (c[1] + 2c[2]) h + \frac{1}{24} c[2] (c[1] + c[2]) h^2 + \right. \\ & \quad \frac{1}{720} (-c[1]^3 + c[1]^2 c[2] + 9c[1] c[2]^2 + 6c[2]^3) h^3 + \\ & \quad \frac{c[2] (-c[1]^3 + c[1]^2 c[2] + 4c[1] c[2]^2 + 2c[2]^3) h^4}{1440} + \frac{1}{30240} \\ & \quad (c[1]^5 - c[1]^4 c[2] - 6c[1]^3 c[2]^2 + 6c[1]^2 c[2]^3 + 15c[1] c[2]^4 + 6c[2]^5) h^5 + \frac{1}{120960} \\ & \quad c[2] (2c[1]^5 - 2c[1]^4 c[2] - 5c[1]^3 c[2]^2 + 5c[1]^2 c[2]^3 + 9c[1] c[2]^4 + 3c[2]^5) h^6 + \\ & \quad \frac{1}{3628800} (-3c[1]^7 + 3c[1]^6 c[2] + 17c[1]^5 c[2]^2 - 17c[1]^4 c[2]^3 - \\ & \quad 25c[1]^3 c[2]^4 + 25c[1]^2 c[2]^5 + 35c[1] c[2]^6 + 10c[2]^7) h^7 + \\ & \quad \frac{1}{7257600} c[2] (-3c[1]^7 + 3c[1]^6 c[2] + 7c[1]^5 c[2]^2 - 7c[1]^4 c[2]^3 - \\ & \quad 7c[1]^3 c[2]^4 + 7c[1]^2 c[2]^5 + 8c[1] c[2]^6 + 2c[2]^7) h^8 + \\ & \quad \left. (c[2]^9 \gamma_9 + c[1] (c[2]^8 \gamma_{18} + c[1] (c[2]^7 \gamma_{27} + c[1] c[2]^6 \gamma_{36} + c[1]^2 c[2]^5 \gamma_{45} + c[1]^3 c[2]^4 \gamma_{54} + \right. \\ & \quad \left. c[1]^4 c[2]^3 \gamma_{63} + c[1]^5 c[2]^2 \gamma_{72} + c[1]^6 c[2] \gamma_{81} + c[1]^7 \gamma_{90})) \right\} h^9 + O[h]^{10} \end{aligned}$$

`Series[ysol /. c[i_] => hc[i], {h, 0, 6}]`

$$\begin{aligned} & \frac{1}{2} + \frac{1}{12} (c[1] + 2c[2]) h + \frac{1}{24} c[2] (c[1] + c[2]) h^2 + \\ & \quad \frac{1}{720} (-c[1]^3 + c[1]^2 c[2] + 9c[1] c[2]^2 + 6c[2]^3) h^3 + \\ & \quad \frac{c[2] (-c[1]^3 + c[1]^2 c[2] + 4c[1] c[2]^2 + 2c[2]^3) h^4}{1440} + \frac{1}{30240} \\ & \quad (c[1]^5 - c[1]^4 c[2] - 6c[1]^3 c[2]^2 + 6c[1]^2 c[2]^3 + 15c[1] c[2]^4 + 6c[2]^5) h^5 + \frac{1}{120960} \\ & \quad c[2] (2c[1]^5 - 2c[1]^4 c[2] - 5c[1]^3 c[2]^2 + 5c[1]^2 c[2]^3 + 9c[1] c[2]^4 + 3c[2]^5) h^6 + O[h]^7 \end{aligned}$$

`vars`

`{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{16}, \gamma_{17}, \gamma_{18},`
`\gamma_{20}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}, \gamma_{26}, \gamma_{27}, \gamma_{30}, \gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}, \gamma_{35}, \gamma_{36}, \gamma_{40}, \gamma_{41}, \gamma_{42},`
`\gamma_{43}, \gamma_{44}, \gamma_{45}, \gamma_{50}, \gamma_{51}, \gamma_{52}, \gamma_{53}, \gamma_{54}, \gamma_{60}, \gamma_{61}, \gamma_{62}, \gamma_{63}, \gamma_{70}, \gamma_{71}, \gamma_{72}, \gamma_{80}, \gamma_{81}, \gamma_{90}}`

$$\frac{1}{6} c[2] (2c[1] + c[2] - 3(c[1] + c[2]) \gamma_0 + 6c[2] \gamma_0^2 - 6c[2] \gamma_1 - 6c[1] \gamma_{10}) /.$$

$$\left\{ \gamma_0 \rightarrow \frac{1}{2}, \gamma_{10} \rightarrow \frac{c[1] + 2c[2] - 12c[2] \gamma_1}{12c[1]} \right\} // \text{Simplify}$$

0

`Simplify[eqns2 /. psol]`

`{{True, True}}`